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On entanglement of formation

Adam W Majewski

Institute of Theoretical Physics and Astrophysics, University of Gdańsk, Wita Stwosza 57, PL 80-952 Gdańsk, Poland

E-mail: fizwam@univ.gda.pl

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Abstract

In this paper we present the novel qualities of entanglement of formation (EoF) for general (so also infinite-dimensional) quantum systems. A major benefit of our presentation is a rigorous description of EoF. In particular, we indicate how this description may be used to examine optimal decompositions. Illustrative examples showing the method of estimation of EoF are given.

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1. Introduction

The problem of quantum entanglement of mixed states has attracted much attention recently and it has been widely considered in different physical contexts (cf [2] and references therein, see also [3–6]). Following recent works by Peres [1] and Horodecki *et al* [2, 7] there exists a simple criterion allowing one to judge whether a given density matrix ϱ, representing a 2×2 or 2×3 composite system, is separable. On the other hand, the definition of a measure of entanglement for general quantum systems, as well as the problem of finding operational, sufficient and necessary conditions for separability in higher dimensions, still remains open (cf [8, 9], and [2] and references therein).

In this paper we are concerned with the entanglement of formation, EoF, introduced in [10]. Let us stress that the principal motivation for our generalization of the definition of EoF follows from the foundations of quantum mechanics: a quantum system is described by infinite-dimensional Hilbert space. Further, to indicate that this concept stems from the mathematical structure of a tensor product we develop the theory of EoF in general terms of composite systems. Moreover, we look more closely at the original definition of EoF. Namely, there is a difficulty in implementing the definition given by Bennett *et al*, in the sense that it is not clear why the operation of taking *min* over the set of all decompositions of the given state into a finite convex combination of pure states is well defined (for details see *Optimal decompositions* in section 5.3). To overcome this problem and to have a measure with nice topological properties we shall use the theory of decomposition which is based on the theory of compact convex sets and boundary integrals. This paper is organized as follows. In section 2 we set up the notation and terminology, and we review some of the standard facts on the theory of decomposition. Section 3 contains our definition of EoF with the proof that, according to our definition, EoF is equal to zero if and only if a state is a separable one. In other words, EoF leads to a well established criterion of separability. In section 4 we review the properties of EoF. Namely, we indicate how techniques based on decomposition theory can be used to study EoF. Moreover, some simple examples showing the method of estimation of EoF are given. Furthermore, the proof of convexity and a detailed study of the topological properties of EoF are obtained. In particular, it is shown that *the family of maximally entangled states is a subset of pure states*. In the final section 5, we present some other examples of explicitly calculated EoF and we clarify the relation between our definition and that of Bennett *et al*. Furthermore, we provide a detailed exposition of the concept of optimal decompositions. Also, some remarks concerning the uniqueness of the measure of entanglement are given.

2. Preliminaries

Let us consider a composite system '1 + 2' and its Hilbert space of the pure states $\mathcal{H}_1 \otimes \mathcal{H}_2$ where $\mathcal{H}_i$ is the Hilbert space associated with subsystem i ($i = 1, 2$). Let $\mathcal{B}(\mathcal{H}_1)$ denote the set of all bounded linear operators on $\mathcal{H}_1$ while $\mathcal{M}$ stands for a (unital) C^*-subalgebra of $\mathcal{B}(\mathcal{H}_2)$. We will assume that $\mathcal{H}_1$ is finite-dimensional space (in our concluding remarks, section 5, we will indicate how to dispense with that assumption). $\mathcal{H}_2$ will be an arbitrary (infinite-dimensional, separable) Hilbert space. In other words, the composite system consists of a small subsystem and a large heat bath, a rather typical situation for concrete physical problems. Any density matrix (normal state) on $\mathcal{H}$ determines uniquely a linear positive, normalized, functional $\omega_\varrho(\cdot) \equiv \omega(\cdot) \equiv \mathrm{Tr}\{\varrho\cdot\}$ on $\mathcal{B}(\mathcal{H})$ which is also called a state.

We will assume the Ruelle separability condition for $\mathcal{M}$ (cf [11, 12, 15]): a subset $\mathcal{F}$ of the set of all states $\mathcal{S}$ of $\mathcal{M}$ satisfies the separability condition if there exists a sequence $\{\mathcal{M}_n\}$ of sub-C^*-algebras of $\mathcal{M}$ such that $\cup_{n\geqslant 1}\mathcal{M}_n$ is dense in $\mathcal{M}$, and each $\mathcal{M}_n$ contains a closed, two-sided, separable ideal $\mathcal{I}_n$ such that

$$\mathcal{F} = \{\omega; \omega \in \mathcal{S}, \|\omega|_{\mathcal{I}_n}\| = 1, n \geqslant 1\}. \tag{1}$$

This condition leads to a situation in which the subsets of states have good measurability properties. Furthermore, one can verify that this separability condition is satisfied for two important cases:

- $\mathcal{M}$ is a separable C^*-algebra. Then $\mathcal{S}$ is metrizable and the Borel and Baire structures on $\mathcal{S}$ coincide. We put $\mathcal{F} = \mathcal{S}$ in that case.
- $\mathcal{M} = \mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and $\mathcal{F}$ is the set of all density matrices (normal states).

Thus, the separability condition covers the basic models of quantum mechanics and we will restrict our attention to models satisfying this condition. However, generalizations of our approach are possible.

We recall that the density matrix ϱ (state) on the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$ is called separable if it can be written or approximated (in the norm) by the density matrices (states) of the form:

$$\varrho = \sum p_i \varrho_i^1 \otimes \varrho_i^2 \qquad \left(\omega(\cdot) = \sum p_i (\omega_i^1 \otimes \omega_i^2)(\cdot)\right)$$

where $p_i \geqslant 0$, $\sum_i p_i = 1$, ϱ_i^α are density matrices on $\mathcal{H}_\alpha$, $\alpha = 1, 2$, and $(\omega_i^1 \otimes \omega_i^2)(A \otimes B) \equiv \omega_i^1(A) \cdot \omega_i^2(B) \equiv (\mathrm{Tr}\, \varrho_i^1 A) \cdot (\mathrm{Tr}\, \varrho_i^2 B) \equiv \mathrm{Tr}\{\varrho_i^1 \otimes \varrho_i^2 \cdot A \otimes B\}$.

Now, for the convenience of the reader, we introduce some terminology and give a short resume of results from convexity and Choquet theory that we shall need in the following (for details see [16, 17, 22], and [15]). Let $\mathcal{A}$ stand for a C^*-algebra. From now on we make the same assumption of separability for $\mathcal{A}$ which was posed for $\mathcal{M}$. In the next sections, by a slight abuse of notation we will write $\mathcal{A}$ for $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$. By $\mathcal{S}$ we will denote the state space of $\mathcal{A}$, i.e. the set of linear, positive, normalized, linear functionals on $\mathcal{A}$. We recall that $\mathcal{S}$ is a compact convex set in the *-weak topology. Further, we denote by $M_1(\mathcal{S})$ the set of all probability Radon measures on $\mathcal{S}$. It is well known that $M_1(\mathcal{S})$ is a compact subset of the vector space of real, regular Borel measures on $\mathcal{S}$. After these preliminaries let us recall the concept of a barycentre $b(\mu)$ of a measure $\mu \in M_1(\mathcal{S})$:

$$b(\mu) = \int \mathrm{d}\mu\,(\varphi)\varphi \tag{2}$$

where the integral is understood in the weak sense. The set $M_\omega(\mathcal{S})$ is defined as a subset of $M_1(\mathcal{S})$ with barycentre ω, i.e.

$$M_\omega(\mathcal{S}) = \{\mu \in M_1(\mathcal{S}), b(\mu) = \omega\}. \tag{3}$$

$M_\omega(\mathcal{S})$ is a convex closed subset of $M_1(\mathcal{S})$, hence compact in the weak *-topology. Hence, it follows by the Krein–Milman theorem that there are 'many' extreme points in $M_\omega(\mathcal{S})$. We say the measure μ is simplicial if μ is an extreme point in $M_\omega(\mathcal{S})$. We denote by $\mathcal{E}_\omega(\mathcal{S})$ the set of all simplicial measures in $M_\omega(\mathcal{S})$. Finally, we will need the concept of orthogonal measures. To define that concept one introduces firstly the notion of orthogonality of positive linear functionals on $\mathcal{A}$: given positive functionals ϕ, ψ on $\mathcal{A}$ we say that ϕ and ψ are orthogonal, in symbols $\phi \perp \psi$, if for all positive linear functionals γ on $\mathcal{A}$, $\gamma \leqslant \phi$ and $\gamma \leqslant \psi$ imply that $\gamma = 0$.

Turning to measures, let μ be a regular non-negative Borel measure on $\mathcal{S}$ and let μ_V denote the restriction of μ to V for a measurable set V in $\mathcal{S}$, i.e. $\mu_V(T) = \mu(V \cap T)$ for T measurable in $\mathcal{S}$. If for all Borel sets V in $\mathcal{S}$ we have

$$\int_{\mathcal{S}} \varphi\, \mathrm{d}\mu_V\,(\varphi) \perp \int_{\mathcal{S}} \varphi\, \mathrm{d}\mu_{\mathcal{S}\setminus V}\,(\varphi) \tag{4}$$

we say that μ is an orthogonal measure on $\mathcal{S}$. We recall that the set of all orthogonal measures on $\mathcal{S}$ with barycentre ω, $O_\omega(\mathcal{S})$, forms a subset (in general proper) of $\mathcal{E}_\omega(\mathcal{S})$, i.e. $O_\omega(\mathcal{S}) \subset \mathcal{E}_\omega(\mathcal{S})$.

3. Entanglement of formation

Let us define, for a state ω on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$, the following map:

$$(r\omega)(A) \equiv \omega(A \otimes \mathbf{1}) \tag{5}$$

where $A \in \mathcal{B}(\mathcal{H}_1)$.

Clearly, $r\omega$ is a state on $\mathcal{B}(\mathcal{H}_1)$. One has the following statement:

Let $(r\omega)$ be a pure state on $\mathcal{B}(\mathcal{H}_1)$ (so a state determined by a vector from $\mathcal{H}_1$). Then ω can be written as a product state on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$.

The proof of this statement can be extracted from [13]. However, for the convenience of the reader we provide the basic idea of the proof. It is enough to consider the case with an arbitrary but fixed positive B in unit ball of $\mathcal{M}$ such that $0 < \omega(\mathbf{1} \otimes B) < 1$. Then $(r\omega)(A)$ can be written as

$$(r\omega)(A) = \omega(\mathbf{1} \otimes B)\omega^I(A) + (1 - \omega(\mathbf{1} \otimes B))\omega^{II}(A) \tag{6}$$

where $\omega^I(A) = \frac{1}{\omega(\mathbf{1}\otimes B)}\omega(A \otimes B)$ and $\omega^{II}(A) = \frac{1}{1-\omega(\mathbf{1}\otimes B)}\omega(A \otimes (\mathbf{1} - B))$. Clearly ω^I and ω^{II} are well defined linear, positive functionals (states) on $\mathcal{B}(\mathcal{H}_1)$. Hence, the purity of $(r\omega)$

implies $\omega^I = \omega^{II}$. Consequently, $\omega(A \otimes B) = \omega(A \otimes \mathbf{1})\omega(\mathbf{1} \otimes B)$. The rest is straightforward so the proof is completed. (For more details we refer the reader to [13, 14].)

Conversely, there is another result in operator algebras saying that, if ω is a state on $\mathcal{B}(\mathcal{H}_1)$, then there exists a state ω' over $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ which extends ω. If ω is a pure state of $\mathcal{B}(\mathcal{H}_1)$ then ω' may be chosen to be a pure state of $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ (cf [15]).

Now we are in a position to give a modification and discuss the definition of EoF (cf [10]).

Let ω be a state on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$. *The EoF is defined as*

$$E(\omega) = \inf_{\mu \in M_\omega(\mathcal{S})} \int_{\mathcal{S}} \mathrm{d}\mu\,(\varphi) S(r\varphi) \tag{7}$$

where $S(\cdot)$ *stands for the von Neumann entropy, i.e.* $S(\varphi) = -\operatorname{Tr} \varrho_\varphi \log \varrho_\varphi$ *where* ϱ_φ *is the density matrix determining the state* φ*.*

In order to comment on the above definition we recall that the map r and the function S are (*-weakly) continuous. At this point we want to strongly emphasize that we use the entropy function S only to respect the tradition. Namely, to have a well defined concept of EoF we need a concave non-negative continuous function which vanishes on pure states (and only on pure states). In our case, with the first subsystem being finite, the von Neumann entropy meets these conditions. Thus, we define EoF as an infimum of integrals evaluated on a continuous function and the infimum is taken over the compact set. Therefore, the infimum is attainable, i.e. there exists a measure $\mu_0 \in M_\omega(\mathcal{S})$ such that

$$E(\omega) = \int_{\mathcal{S}} \mathrm{d}\mu_0\,(\varphi) S(r\varphi) \tag{8}$$

and

$$\omega = \int_{\mathcal{S}} \mathrm{d}\mu_0\,(\varphi)\varphi. \tag{9}$$

Now we want to show that $\mathcal{F} \ni \omega \mapsto E(\omega)$ is equal to 0 only for separable states (we recall that $\mathcal{F}$ stands for the subset of states satisfying Ruelle's condition, cf section 2). Assume $E(\omega) = 0$. Then

$$\int_{\mathcal{S}} \mathrm{d}\mu_0\,(\varphi) S(r\varphi) = 0 \tag{10}$$

for some probability measure μ_0. As $S(r\varphi) \geqslant 0$ and it is the continuous function we infer that $S(r\varphi) = 0$ for each φ in the support of μ_0. But, as the entropy is a concave function we have

$$S \circ r(\varphi) \geqslant \int \mathrm{d}\xi\,(\nu) S(r\nu) \tag{11}$$

for any positive measure $\mathrm{d}\xi$ on $\mathcal{S}$ such that $\varphi = \int \mathrm{d}\xi\,(\nu)\nu$. In particular, taking a measure supported on pure states (such a decomposition always exists under the assumed separability condition) we infer $S(r\nu) = 0$, so $r\nu$ is a pure state and consequently ν is a product state. So φ is a convex combination of product states. Finally, as $\omega = \int_{\mathcal{S}} \mathrm{d}\mu_0\,(\varphi)\varphi$ and μ_0 can be well approximated by finite measures (see [18]), we infer that ω can be approximated by convex combinations of product states, so ω is a separable state.

Now, let us assume that ω is a separable state, i.e. ω can be approximated by convex combinations of product states $\omega_i^{(N)}$:

$$\omega = \lim_N \sum_{i=1}^{N} \lambda_i^{(N)} \omega_i^{(N)}. \tag{12}$$

Define

$$\mu^N = \sum_{i=1}^{N} \lambda_i^{(N)} \delta_{\omega_i^{(N)}} \tag{13}$$

where $\delta_{\omega_i^{(N)}}$ are the Dirac measures of the point $\omega_i^{(N)}$. Considering the weak limit of $\int \mathrm{d}\mu^N(\varphi)\varphi$ we can infer that there is a measure μ such that

$$\int \mathrm{d}\mu\,(\varphi)\varphi = \omega \qquad \int \mathrm{d}\mu\,(\varphi)S(r\varphi) = 0. \tag{14}$$

So we arrive at:

Theorem 1. *A state $\omega \in \mathcal{F}$ is separable if and only if EoF $E(\omega)$ is equal to 0.*

4. Properties of EoF

4.1. Relations to other decompositions

Let us discuss some relations between decompositions used in our definition of EoF and other types of decompositions. Assume that the state ω is separable, so there is a measure $\mu_0 \in M_\omega(\mathcal{S})$ such that $\int \mathrm{d}\mu_0\,(\varphi)S(r\varphi) = 0$. But as we consider a non-negative function and positive measures this implies that there is a simplicial measure μ_0^s (in fact, there can be many such measures) such that $\int \mathrm{d}\mu_0^s\,(\varphi)S(r\varphi) = 0$. In other words, the infimum is attainable on the set of simplicial measures $\mathcal{E}_\omega(\mathcal{S})$ (for a more detailed discussion on the role of simplicial measures in the description of EoF see section 5.3).

On the other hand, as $O_\omega(\mathcal{S}) \subset \mathcal{E}_\omega(\mathcal{S})$ we have

$$\inf_{\mu\in\mathcal{E}_\omega(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi)S(r\varphi) \leqslant \inf_{\mu\in O_\omega(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi)S(r\varphi). \tag{15}$$

In general we cannot expect the equality in (15). Namely, there are examples of simplicial measures which are not orthogonal (cf [19]). So finding an orthogonal measure such that 'inf' is attained we can infer that the state is separable but not conversely. To be clearer, let us recall some algebraic aspects of decomposition theory (cf [15]) which are related to orthogonal measures. A finite convex decomposition of $\omega \in \mathcal{S}$ corresponds to a finite decomposition of identity $\mathbf{1} = \sum_i T_i$, $T_i \geqslant 0$ within the commutant $\pi_\omega(\mathcal{A})'$. The simplest form of such a decomposition occurs when the T_i are mutually orthogonal projections. This type of decomposition corresponds to that determined by orthogonal measure. So, taking the spectral resolution of density matrix ϱ_ω we obtain the very special (subcentral) orthogonal decomposition. Therefore, if we restrict ourselves to decomposition induced by spectral resolution of ϱ_ω, in general, we cannot expect to attain $\inf_{\mu\in M_\omega(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi)S(r\varphi)$, see also section 5.3.

4.2. Examples: I

To illustrate the question of computation of EoF we start with very simple models.

(1) The von Neumann entropy (for finite systems) is maximal for the state of the form $\omega_{\varrho_m}(A) = \mathrm{Tr}\,\varrho_m A$ with $\varrho_m = \frac{1}{\dim\mathcal{H}}\mathbf{1}$ (dim stands for dimension). For such state it is equal to $\ln(\dim\mathcal{H})$ and this is the maximal value of E.

(2) Let us consider 2×2 system with $\mathcal{H}_1 \equiv \mathcal{H}_2$ (so $\dim\mathcal{H}_1 = 2 = \dim\mathcal{H}_2$) and the singled state Ψ_- defined as $|\Psi_-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$. Here we adopt a notation of quantum mechanics by writing $|01\rangle \equiv e_0 \otimes e_1$, where $\{e_0, e_1\}$ is a basis in $\mathcal{H}_1$, etc. Write $\omega_{\Psi_-}(A) = \mathrm{Tr}\{|\Psi_-\rangle\langle\Psi_-| \cdot A \otimes B\}$ where $A \in \mathcal{B}(\mathcal{H}_1)$ while $B \in \mathcal{B}(\mathcal{H}_2)$. Then $r\omega_{\Psi_-}(A) = \mathrm{Tr}\{|\Psi_-\rangle\langle\Psi_-| \cdot A \otimes \mathbf{1}\} = \mathrm{Tr}\{(\frac{1}{2}\mathbf{1})A\}$. So $E(\omega_{\Psi_-}) = \ln 2$.

(3) Let us consider a $d \times d$ system and a so-called maximally entangled state $|\Psi_+^d\rangle = \frac{1}{\sqrt{d}}\sum_{i=1}^d |i\rangle \otimes |i\rangle$ where $\{|i\rangle\}$ is a basis in $\mathcal{H}_1 = \mathcal{H} = \mathcal{H}_2$. Again, let us define

$\omega_{\Psi_+^d}(A \otimes B) = \mathrm{Tr}\{|\Psi_+^d\rangle\langle\Psi_+^d| A \otimes B\}$ and consider $r\omega_{\Psi_+^d}$. It is easy to note that $r\omega_{\Psi_+^d}(A) = \mathrm{Tr}\{(\frac{1}{d}\mathbf{1})A\}$. Hence $E(\omega_{\Psi_+^d}) = \ln d$, so E attains its maximal value.

The results just listed are easy to show since there is no question concerning the non-uniqueness of the decomposition of the (pure) state ω into pure states.

4.3. Convexity of EoF

To prove the convexity of EoF let us show that the set $M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathcal{S})$ contains the sum of the sets $\lambda_1 M_{\omega_1}(\mathcal{S})$ and $\lambda_2 M_{\omega_2}(\mathcal{S})$, where λ_1 and λ_2 are non-negative numbers such that $\lambda_1+\lambda_2 = 1$. To see this we recall (see, e.g., [15] or [21]) that $\mu \in M_\omega(\mathcal{S})$ if and only if $\mu(f) \geqslant f(\omega)$ for any continuous, real-valued, convex function f. Thus

$$(\lambda_1\mu_1 + \lambda_2\mu_2)(f) \geqslant \lambda_1 f(\omega_1) + \lambda_2 f(\omega_2) \geqslant f(\lambda_1\omega_1 + \lambda_2\omega_2) \tag{16}$$

implies the above stated relation between sets. Hence

$$\begin{aligned} E(\lambda_1\omega_1 + \lambda_2\omega_2) &= \inf_{\mu\in M_{\lambda_1\omega_1+\lambda_2\omega_2}(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi) S(r\varphi) \\ &\leqslant \lambda_1 \inf_{\mu\in M_{\omega_1}(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi) S(r\varphi) + \lambda_2 \inf_{\mu\in M_{\omega_2}(\mathcal{S})} \int \mathrm{d}\mu\,(\varphi) S(r\varphi) \\ &= \lambda_1 E(\omega_1) + \lambda_2 E(\omega_2). \end{aligned} \tag{17}$$

Consequently, the function $\mathcal{S} \ni \omega \mapsto E(\omega)$ is the convex one.

4.4. Subadditivity

Consider the tensor product of von Neumann algebras $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \otimes \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$ and a state $\omega \otimes \omega$ over it where ω is a state on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$. We observe

$$\begin{aligned} E(\omega\otimes\omega) &= \inf_{\mu\in M_{\omega\otimes\omega}(\mathcal{S}_T)} \int \mathrm{d}\mu\,(\nu) S_{1+2}(r\nu) \\ &\leqslant \inf_{\mu_1\times\mu_2\in M_\omega(\mathcal{S})\times M_\omega(\mathcal{S})} \int \mathrm{d}\mu_1\,(\nu) \int \mathrm{d}\mu_2\,(\nu') S_{1+2}(r \circ \nu \otimes \nu') \\ &\leqslant \inf_{\mu_1\times\mu_2\in M_\omega(\mathcal{S})\times M_\omega(\mathcal{S})} \int \mathrm{d}\mu_1\,(\nu) \int \mathrm{d}\mu_2\,(\nu') (S_1(r\nu) + S_1(r\nu')) \\ &= 2E(\omega) \end{aligned} \tag{18}$$

where $\mathcal{S}_T$ denotes the set of all states on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M} \otimes \mathcal{B}(\mathcal{H}_1) \otimes \mathcal{M}$, S_{1+2} (S_1) the von Neumann entropy on $\mathcal{B}(\mathcal{H}_1) \otimes \mathcal{B}(\mathcal{H}_1)$ ($\mathcal{B}(\mathcal{H}_1)$ respectively). The last inequality follows from subadditivity of the von Neumann entropy. Consequently, EoF also has a form of subadditivity. Applying the above argument to $E(\omega\otimes\cdots\otimes\omega)$ one can consider the 'density' of EoF and treat $E(\omega)$ as an extensive (thermodynamical) quantity. This feature of EoF seems to be important in quantum information (cf [2]).

4.5. Topological properties of EoF

Now we wish to examine the question of continuity of EoF. To describe that topological property we shall need some preliminaries. Let us consider $M_1(\mathcal{S})$ and $\mathcal{S}$ as two compact spaces and a continuous mapping b of $M_1(\mathcal{S})$ onto $\mathcal{S}$ given by $M_1(\mathcal{S}) \ni \mu \mapsto b(\mu) = \int_{\mathcal{S}} \nu \mathrm{d}\mu\,(\nu)$, so $b(\mu)$ is the barycentre of the measure μ. Moreover, let us consider the equivalence relation $E(b)$ on the set $M_1(\mathcal{S})$ determined by the decomposition $\{b^{-1}(\omega)\}_{\omega\in\mathcal{S}}$ of $M_1(\mathcal{S})$ into fibres of b. We denote by q the mapping of $M_1(\mathcal{S})$ to $M_1(\mathcal{S})/E(b)$, assigning to the point $\mu \in M_1(\mathcal{S})$ the equivalence class $[\mu] \in M_1(\mathcal{S})/E(b)$. We equip $M_1(\mathcal{S})/E(b)$ with the quotient topology, so q is the natural (quotient) mapping. As b is a continuous mapping of the compact (Hausdorff) space $M_1(\mathcal{S})$ to the compact (Hausdorff) space $\mathcal{S}$ then the equivalence relation $E(b)$ is closed. We wish to represent the mapping $b : M_1(\mathcal{S}) \to \mathcal{S}$ as the composition $\overline{b} \circ q$ of the natural mapping q with the mapping $\overline{b}$ of the quotient space $M_1(\mathcal{S})/E(b)$ onto $\mathcal{S}$ defined by letting $\overline{b}(b^{-1}(\omega)) = \omega$. It is an easy observation that the mapping $\overline{b}$ is continuous. Hence we have

$$\begin{tikzcd} M_1(\mathcal{S}) \arrow[rr, "b"] \arrow[dr, "q"'] & & \mathcal{S} \\ & M_1(\mathcal{S})/E(b) \arrow[ur, "\overline{b}"'] & \end{tikzcd}$$

In particular, $\overline{b}$ is one-to-one continuous mapping of $M_1(\mathcal{S})/E(b)$ onto $\mathcal{S}$. We want to show that $\overline{b}$ is a homeomorphism. To prove this we observe that q is the continuous mapping carrying the compact topological space $M_1(\mathcal{S})$ onto the topological space $M_1(\mathcal{S})/E(b)$. Then, $M_1(\mathcal{S})/E(b)$ is pre-compact (not necessary Hausdorff space). But, then $\overline{b}$ is the continuous mapping of the pre-compact space $M_1(\mathcal{S})/E(b)$ onto compact (Hausdorff) space $\mathcal{S}$. Thus, $\overline{b}$ is a homeomorphism. Therefore, we arrive at:

Proposition 1. *$\overline{b}$ is a homeomorphism, i.e. the mapping b is quotient.*

Now we wish to describe equivalence classes in $M_1(\mathcal{S})/E(b)$. Let $\omega_\alpha \in \mathcal{S}$. We observe

$$b^{-1}(\omega_\alpha) = \left\{ \mu \in M_1(\mathcal{S});\ \int \nu \, \mathrm{d}\mu\,(\nu) = \omega_\alpha \right\}. \tag{19}$$

In other words, $b^{-1}(\omega_\alpha)$ is equal to the set $M_{\omega_\alpha}(\mathcal{S})$ of all probabilistic measures on $\mathcal{S}$ which represent the point $\omega_\alpha \subset \mathcal{S}$. We recall (see [21]) that $\mu \in M_{\omega_\alpha}(\mathcal{S})$ is equivalent to $\mu \sim \delta_{\omega_\alpha}$, i.e. that μ is equivalent to the Dirac measure δ_{ω_α} where the equivalence of (probabilistic) measures μ and μ' is defined as

$$\int \nu \, \mathrm{d}\mu\,(\nu) = \int \nu \, \mathrm{d}\mu'\,(\nu). \tag{20}$$

But $\delta_{\omega_\alpha} \sim \mu$, in turn, is equivalent to (cf [21]) $\delta_{\omega_\alpha} - \mu \in \mathcal{N}(\mathcal{S})$ where $\mathcal{N}(\mathcal{S})$ is the annihilator of $A(\mathcal{S})$ in the dual pair $\langle C_{\mathbb{R}}(\mathcal{S}), M_{\mathbb{R}}(\mathcal{S})\rangle$. Here, $A(\mathcal{S})$ $(C_{\mathbb{R}}(\mathcal{S}), M_{\mathbb{R}}(\mathcal{S}))$ is the set of all continuous real-valued affine functions on $\mathcal{S}$ (the vector space of all real-valued continuous functions, the vector space of all real measures on $\mathcal{S}$, respectively). Now it should be clear that an equivalence class $[\mu]$ in $M_1(\mathcal{S})/E(b)$ is equal to $\{\delta_\omega + \mu;\ \mu \in \mathcal{N}(\mathcal{S}),\ \mu(f) \geqslant -f(\omega)$ *for any* $0 \leqslant f \in C_{\mathbb{R}}(\mathcal{S})\}$ with some fixed $\omega \in \mathcal{S}$.

In order to complete our discussion of the diagram we should examine topological properties of the set-valued map b^{-1}.

Proposition 2. *The set-valued map b^{-1} is upper semicontinuous.*

Proof. The map b^{-1} is lower (upper) semicontinuous (cf [20]) if and only if, for every closed set $K \subset M_1(\mathcal{S})$, the set $\mathcal{O}_L = \{\omega : b^{-1}(\omega) \subset K\}$ (the set $\mathcal{O}_U = \{\omega : b^{-1}(\omega) \cap K \neq \emptyset\}$) is closed in $M_1(\mathcal{S})$. To examine upper semicontinuity of b^{-1} let us consider a net $\{\omega_\alpha\} \subset \mathcal{O}_U$ with a limit ω_0. Futhermore, let us choose measures $\{\mu_\alpha \in b^{-1}(\omega_\alpha) \cap K\}$. As K is a compact subset there exists a convergent subnet $\{\mu_\beta\}$ such that $\mu_\beta \to \mu \in K$. Since $\int_{\mathcal{S}} \nu \, \mathrm{d}\mu_\beta\,(\nu) = \omega_\beta$ we infer that

$$\omega_\beta \to \omega = \int_{\mathcal{S}} \nu \, \mathrm{d}\mu(\nu). \tag{21}$$

Then, the uniqueness of the limit implies $\omega_0 = \omega$ and the proof of upper semicontinuity of b^{-1} is complete. □

Having fully clarified topological relations among $\mathcal{S}$, $M_1(\mathcal{S})$ and $M_1(\mathcal{S})/E(b)$ we wish to prove:

Proposition 3. *EoF, $\mathcal{S} \ni \omega \mapsto E(\omega)$, is the continuous function.*

Proof. We start with an easy proof of lower semicontinuity of EoF. To this end let $\{\omega_\alpha\}$ be a net with a limit ω and let $E(\omega_\alpha) \leqslant s$, where s is a real number. To show that $E(\omega) \leqslant s$ let us take $\epsilon > 0$ and choose $\mu_\alpha \in M_{\omega_\alpha}(\mathcal{S})$ such that $\mu_\alpha(S \circ r) < s + \epsilon$. As $M_1(\mathcal{S})$ is a compact set there exists a convergent subnet $\{\mu_\beta\}$ with the limit μ_0. Let $\hat{A}$ be an affine, real-valued continuous function on $\mathcal{S}$. Then

$$\hat{A}(\omega_\beta) = \mu_\beta(\hat{A}) \to \mu_0(\hat{A}) \tag{22}$$

and $\hat{A}(\omega_\beta) \to \hat{A}(\omega)$. Thus $\mu_0 \in M_\omega(\mathcal{S})$. Therefore, $s + \epsilon \geqslant \lim \mu_\beta(S \circ r) = \mu_0(S \circ r) \geqslant E(\omega)$. Consequently, $E(\omega) \leqslant s$ and the proof of lower semicontinuity of EoF is complete.

Now, let us consider the question of upper semicontinuity of $E(\omega)$. Again, let ω_α be a net with a limit point ω_0. Take s such that $E(\omega_\alpha) \geqslant s$. We observe that, for any $\mu_\alpha \in M_{\omega_\alpha}(\mathcal{S})$, $\mu_\alpha(S \circ r) \geqslant s$. Again, the use of a convergent subnet $\{\mu_\beta\}$ with a limit μ_0 implies that $s \leqslant \lim_\beta \mu_\beta(S \circ r) = \mu_0(S \circ r)$, where $\mu_0 \in M_{\omega_0}(\mathcal{S})$. Thus, to prove upper semicontinuity of $E(\omega)$ it is enough to show that any $\mu \in M_{\omega_0}(\mathcal{S})$ can be obtained as a limit of $\{\mu'_\beta\}$, i.e. $M_{\omega_0}(\mathcal{S}) \ni \mu = \lim_\beta \mu'_\beta$ where $\mu'_\beta \in M_{\omega_\beta}(\mathcal{S})$. In particular, we want to have

$$\wedge_{\epsilon>0} \wedge_{\mu \in M_{\omega_0}(\mathcal{S})} \vee_{\mu_\beta \in M_{\omega_\beta}} \qquad |\mu(f) - \mu_\beta(f)| < \epsilon \tag{23}$$

for a continuous function f on $\mathcal{S}$. Assume the contrary, i.e.

$$\vee_{\epsilon>0} \vee_{\mu \in M_{\omega_0}(\mathcal{S})} \wedge_{\mu_\beta \in M_{\omega_\beta}} \qquad |\mu(f) - \mu_\beta(f)| \geqslant \epsilon \tag{24}$$

for a continuous function f. We note (cf [20]) that the upper semicontinuity of b^{-1} implies that, for every open set $U \subset M_1(\mathcal{S})$, the set $\{\omega : b^{-1}(\omega) \subset U\}$ is open. Now, assuming (24) one can find the neighbourhood U_μ of μ which does not contain any μ_β. But $\{\omega : b^{-1}(\omega) \subset U_\mu\}$ is a neighbourhood of ω_0. The convergence $\omega_\beta \to \omega_0$ implies that each neighbourhood of ω_0 should contain states of the form ω_β. Thus, also, $\{\omega : b^{-1}(\omega) \subset U_\mu\}$ should contain many ω_β. Hence, one can find $M_{\omega_\beta}(\mathcal{S}) \subset U_\mu$. But this contradicts (24). Therefore, (23) holds and the proof of upper continuity of $E(\omega)$ is complete. This completes the proof of the proposition. □

As $\mathcal{S} \ni \omega \mapsto E(\omega)$ is a continuous convex function, the application of the Bauer maximum principle leads to:

Corollary 1. *$E(\omega)$ attains its maximum at an extremal point of $\mathcal{S}$, so the family of maximally entangled states is a subset of pure states.*

5. Examples and concluding remarks

5.1. Examples: II

Let us begin this final section with some other illustrative examples showing the usefulness of EoF.

(1) Let $\omega = \sum_k \omega_k$ be a decomposition of the state ω. Then, an application of convexity would lead to the following estimation of entanglement of ω: $E(\omega) \leqslant \max_k E(\omega_k)$.

(2) In the discussion of entangled states the positive partial transposition criterion plays an important role [1, 7]. To consider that question, let us put $\mathcal{M} = \mathcal{B}(\mathcal{H}_2)$ with finite-dimensional Hilbert space $\mathcal{H}_2$. We recall that the map $\alpha : \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2) \to \mathcal{B}(\mathcal{H}_1 \otimes \mathcal{H}_2)$ with $\alpha = \mathrm{i}d \otimes \tau$, where τ is a transposition map of the matrix representation of an arbitrary $B \in \mathcal{B}(\mathcal{H}_2)$ in a certain fixed basis, provides an essential ingredient of that criterion. Let us define $(\alpha^d \omega)(A \otimes B) \equiv \omega(\alpha(A \otimes B))$ and let us note

$$\begin{aligned}(r\alpha^d \omega)(A) &= (\alpha^d \omega)(A \otimes \mathbf{1}) = \omega(\alpha(A \otimes \mathbf{1})) \\ &= \omega(A \otimes \tau(\mathbf{1})) = \omega(A \otimes \mathbf{1}) = (r\omega)(A).\end{aligned} \tag{25}$$

Consequently, the partial transposition does not change the measure of entanglement. Therefore, the basic point of that criterion is that id $\otimes\, \tau$ is not a completely positive map. For further details on the relations between entanglement and positive maps see [14].

(3) Let us consider a $d \times d$ system with the corresponding Hilbert space $\mathcal{H} \equiv \mathcal{H}_1 \otimes \mathcal{H}_2$ and let P be a projector such that $P\mathcal{H}$ does not contain product states. We recall that such projectors are related to the concept of unextendible product bases [23]. Let us define the state ω_P as $\omega_P(A \otimes B) = (\mathrm{Tr}\, P)^{-1}\, \mathrm{Tr}\{P \cdot A \otimes B\}$. We want to judge whether ω_P is a separable state. To this end let us consider an arbitrary decomposition of ω_P, i.e.

$$\begin{aligned}\omega_P(A \otimes B) &= (\mathrm{Tr}\, P)^{-1}\, \mathrm{Tr}\, PA \otimes B \\ &= (\mathrm{Tr}\, P)^{-1} \sum_k \mathrm{Tr}\, Pa_k PA \otimes B \equiv \sum_k \lambda_k \omega_k(A \otimes B)\end{aligned} \tag{26}$$

where operators $a_k \geqslant 0$ are defined on $\mathcal{H}$ and satisfy $\sum_k a_k = \mathbf{1}$ while ω_k stands for the state determined by Pa_kP. Assume ω_P is a separable state. Then, there is a decomposition $\omega_P = \sum_k \lambda_k \omega_k$ such that $r\omega_k$ is a pure state for any k. But, then, ω_k would be a product state, i.e.

$$\omega_k(A \otimes B) = \mathrm{Tr}\, \varrho_k^1 \otimes \varrho_k^2 A \otimes B. \tag{27}$$

This would imply

$$\varrho_k^1 \otimes \varrho_k^2 = \mathrm{constant} \cdot Pa_kP. \tag{28}$$

But this is impossible as $P\mathcal{H}$ does not contain product (vector) states. Consequently, $E(\omega_P) > 0$.

5.2. Comparison with the Bennett, DiVincenzo, Smolin and Wooters definition of EoF

The presented examples with explicitly calculated $E(\omega)$ suggest a close relation of our EoF with the original definition of EoF, given by Bennett *et al*, which will be denoted by EoF_B. Obviously, EoF $\leqslant \mathrm{EoF}_B$. To examine the converse inequality we start with a simple observation that

$$\inf_{\mu \in M_\omega(\mathcal{S})} \int \mathrm{d}\mu(\nu) S(r\nu) = \inf \left\{ \sum_{i=1}^n \lambda_i S(r\nu_i) : \omega = \sum_{i=1}^n \lambda_i \nu_i \quad (\text{convex sum}) \right\} \tag{29}$$

and the first infimum is attained for some $\mu \in M_\omega(\mathcal{S})$. The above observation follows from the fact that each measure μ can be (*-weakly) approximated by measures with finite support. On the other hand, measures concentrated on $\mathcal{S}_p$, where $\mathcal{S}_p$ is the set of all pure states, are known to be maximal with respect to the order $\mu \prec \nu$ ($\mu \prec \nu$ if and only if $\mu(f) \leqslant \nu(f)$ for any convex, real-valued convex function f, cf [16] or [21]), so minimal on the set of all concave functions. In particular, such a measure is minimal on $S \circ r$. Apparently, the maximality of a measure (on all convex functions) is too strong a demand and therefore to get the converse inequality between EoF and EoF_B some extra arguments should be given (see the next subsection).

5.3. Optimal decompositions

In this section we wish to examine the question of the existence of a very special type of decomposition, so-called optimal decompositions. A decomposition $\omega = \sum_{j=1}^{n} \lambda_j \varrho_j$, where $\{\varrho_i\}$ are pure states, such that the infimum in the definition of EoF is attained, will be called *an optimal decomposition*. In other words, the infimum is attained by a measure μ_0 with finite support contained in the set of all pure states. Thus

$$E(\omega) = \inf_{\mu \in M_\omega(\mathcal{S})} \int_{\mathcal{S}} S(r\varrho)\,\mathrm{d}\mu(\varrho) = \int_{\mathcal{S}} \mathrm{d}\mu_0\,(\varrho) S(r\varrho)$$

with supp $\mu_0 = \{\varrho_1, \ldots, \varrho_n\}$, $n < \infty$ and $\varrho_i \in \mathcal{S}_p$. Clearly, $\mu_0 = \sum_1^n \lambda_i \delta_{\varrho_i}$ where δ_ϱ stands for the Dirac measure, $\{\varrho_i\}$ are pure states and $\omega = \sum \lambda_i \varrho_i$. We recall that each measure has maximal balayage and each state $\omega \in \mathcal{S}$ is the barycentre of a measure which is maximal for the order $\succ$ (for all necessary details see [15, 16, 21] or [22]). It is an easy observation that μ_0 is a maximal measure.

In order to avoid misunderstanding we repeat: entropy is concave, maximality is defined for convex functions, so inf for '$x \mapsto -x \ln x$' means sup for '$x \mapsto x \ln x$'.

We recall that maximality of the measure μ_0 implies

$$\mu_0 \text{ is supported by the set } \{\omega; \omega \in \mathcal{S}, f(\omega) = \overline{f}(\omega)\} \tag{30}$$

for all continuous convex functions on $\mathcal{S}$ where $\overline{f}$ is the upper envelope of f, i.e. $\overline{f}(\omega) = \inf\{g(\omega); -g \in P(\mathcal{S}), g \geqslant f\}$ ($P(\mathcal{S})$ stands for the set of continuous convex functions on $\mathcal{S}$). Moreover, if $\overline{f}$ is the upper envelope of a continuous function on $\mathcal{S}$ then

$$\overline{f}(\omega) = \sup_{\mu \in M_\omega(\mathcal{S})} \int f(\nu)\,\mathrm{d}\mu\,(\nu). \tag{31}$$

Denote the set of all maximal measures in $M_\omega(\mathcal{S})$ by $\mathcal{Z}_\omega(\mathcal{S})$. One can show (cf [21]) that $\mathcal{Z}_\omega(\mathcal{S})$ is a face of $M_\omega(\mathcal{S})$. The set of all extremal measures in $\mathcal{Z}_\omega(\mathcal{S})$ will be denoted by $\mathcal{E}_\omega^Z(\mathcal{S})$, i.e.

$$\mathcal{E}_\omega^Z(\mathcal{S}) = \mathcal{E}_\omega(\mathcal{S}) \cap \mathcal{Z}_\omega(\mathcal{S}) \tag{32}$$

where $\mathcal{E}_\omega(\mathcal{S})$ stands for simplicial measures (cf section 4). Let us consider (cf [21])

$$F_0 = \{\mu \in M_\omega(\mathcal{S}); \mu(f) = \overline{f}(\omega)\} \tag{33}$$

where f is a convex continuous function on $\mathcal{S}$. Obviously, by virtue of (31) the set F_0 is not empty. Let $\mu \in F_0$ and assume $\mu = \lambda_1\mu_1 + (1 - \lambda_1)\mu_2$ where $\mu_1, \mu_2 \in M_\omega(\mathcal{S})$ and $\lambda_1 \in (0, 1)$. Clearly, μ_1 and μ_2 should be in F_0. Thus, F_0 is a face. It is an easy observation that F_0 is the closed set with the property: $\mu \in F_0$, $\nu \in M_\omega(\mathcal{S})$ and $\mu \prec \nu$ imply $\nu \in F_0$, i.e. F_0 is the hereditary upwards face. Then, the application of the Lumer existence theorem (see proposition 1.6.4 in [21]) proves the existence of a measure μ_0 in $F_0 \cap \mathcal{E}_\omega^Z(\mathcal{S})$. Thus, we have proved:

Proposition 4. *Maximum of the set $\{\mu(-S \circ r); \mu \in M_\omega(\mathcal{S})\}$ for a continuous convex function $-S$ is attained by a simplicial boundary measure.*

Now, let us examine a very special case. Namely, if the Hilbert space $\mathcal{H}$ of the composite system is of the finite dimension, say $\dim \mathcal{H} = n$, then $\mathcal{S}$ can be considered as a compact subset in the space $\mathbb{R}^{(2n)^2}$. On the other hand, a probability measure μ on a convex compact subset in $\mathbb{R}^{(2n)^2}$ is simplicial if and only if μ is supported by an affinely independent set of (at most $(2n)^2 + 1$) points. Moreover, in the considered (finite-dimensional) case the weak-* topology is metrizable. Hence, maximal measures are supported by extremal points in $\mathcal{S}$. Combining the results just given for the concave function $S \circ r$ we can infer the existence of optimal decomposition. Obviously, it does not exclude a possibility that *infimum* is attainable on some other measures. Nevertheless, for the finite-dimensional case we have $\mathrm{EoF} = \mathrm{EoF}_B$.

Turning to other decompositions (cf section 4.1) we note that, even in the two-dimensional case (so for the algebra of 2×2 matrices), one can write a simplicial maximal measure that is not orthogonal (cf [15]). Therefore, even for a low-dimensional case one cannot say that EoF, $E(\omega) = \inf_{\mu \in \mathcal{O}_\omega(\mathcal{S})} \mu(S \circ r)$.

Finally we want to point out that the proof of the existence of optimal decomposition depends only on the structure of the set $M_\omega(\mathcal{S})$.

5.4. Remarks

There is a frequently considered question of the uniqueness of a measure of entanglement. We already observed (see comments following the definition of EoF in section 3) that one can replace the von Neumann entropy by any continuous non-negative concave function which vanishes on pure states, e.g. $\varrho \mapsto \mathrm{Tr}\{\varrho(\mathbf{1} - \varrho)\}$. Then, all arguments can be repeated and we would arrive at a new measure of entanglement. As a matter of fact, the only reason for our assumption of $\dim \mathcal{H}_1 < \infty$ was that, in this case, the von Neumann entropy is continuous. Thus, to perform a further generalization of EoF one can replace $\varrho \mapsto -\mathrm{Tr}\{\varrho \ln \varrho\}$ by another function from the class just mentioned. Clearly, changing the von Neumann entropy function, the argument leading to the subadditivity of $E(\omega)$ should be modified. The additional reason to use the von Neumann entropy throughout the paper follows from the fact that there is a nice relation (cf [24–26]) between the entropy H of subalgebra $\mathcal{M}_1$ in $\mathcal{M}_1 \otimes \mathcal{M}$ relative to the state ω ($\mathcal{M}_1$, and $\mathcal{M}$ are von Neumann algebras), the von Neumann entropy of the restricted state and EoF:

$$H_{\omega, \mathcal{M}_1 \otimes \mathcal{M}}(\mathcal{M}_1) = S(r\omega) - E(\omega). \tag{34}$$

It is clear that the main difficulty for calculating H is encoded in $E(\omega)$. Therefore, our results concerning EoF shed some new light on the nature of H.

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